Nuclear and Particle Physics
Lecture 4

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Recap puzzle

Concept	Equation	Definition
Exponential decay	$N(t) = N_0e^{-\lambda t}$	Number of nuclei that have decayed in the time *t*
Activity	$A(t) = A_0e^{-\lambda t}$	Number of nuclei decaying per unit time
Decay probability	$P_{decay}(t) = \lambda e^{-\lambda t}$	Probability of a single nucleus decaying in the interval *t*
Lifetime	$\tau = 1/\lambda$	Maximum time until an unstable nucleus decays
Half-life	$t_{1/2} = \ln2/\lambda$	Time by which half the radioactive sample has not yet decayed

Answers

Concept	Equation	Definition
Exponential decay	$N(t) = N_0 e^{-\lambda t}$	Number of nuclei that have not decayed by time *t*
Activity	$A(t) = A_0 e^{-\lambda t}$	Number of nuclei decaying per unit time, where $A_0 = \lambda N_0$
Decay probability	$P_{decay}(t) = \lambda e^{-\lambda t}$	Probability of a single nucleus decaying in the interval $t \rightarrow t+dt$
Mean lifetime or simply *lifetime*	$\tau = 1/\lambda$	Mean time until an unstable nucleus decays
Half-life	$t_{1/2} = \ln 2/\lambda$	Time after which half the radioactive sample has decayed

Simple decay

If a sample of material consists of nucleus A which is unstable and decays to nucleus B (of which there are initially none) we have simply:

$A \rightarrow B$

Nomenclature:
A - “parent”
B - “daughter”

The initial number of each nucleus is:

$N_A(t=0) = N_0$ (= total number of nuclei)

$N_B(t=0) = 0$

As nucleus A decays into nucleus B

$N_A(t) = N_0 e^{-\lambda_A t}$

and since

$N_0 = N_A(t) + N_B(t)$

$N_B(t) = N_0(1 - e^{-\lambda_A t})$

Alternative decay modes

An initial nuclide A that decays into two products: A → B + C

We have at any time *t*: $N_A(t) + N_B(t) + N_C(t) = N_0$ and

$$\frac{dN_A}{dt} = -\lambda_A N_A, \quad \frac{dN_B}{dt} = \lambda_B N_A, \quad \frac{dN_C}{dt} = \lambda_C N_A$$

with $\lambda_A = \lambda_B + \lambda_C$. The decay constants λ_B and λ_C only determine the probabilities of the decays to products B or C

$$N_B(t) = \frac{\lambda_B}{\lambda_A} N_0 (1 - e^{-\lambda_A t})$$

$$N_C(t) = \frac{\lambda_C}{\lambda_A} N_0 (1 - e^{-\lambda_A t})$$

and

$$N_A(t) = N_0 - N_B(t) - N_C(t) = N_0 e^{-\lambda_A t}$$

Decay series (or chains)

Many heavy nuclei decay via complicated series involving several α and β decays. Consider the simple case of A → B → C, where C is stable and only A is present initially:

A → B → C

The number of nuclei A vary according to:

$$N_A(t) = N_o e^{-\lambda_A t}$$

The number of nuclei B as a function of time can be found from:

$$\frac{dN_B(t)}{dt} = -\lambda_B N_B(t) + \lambda_A N_A(t) \qquad (1)$$

where the first term is the decay of nuclei B and the second term is due to B being created from the decay of A.

Integrating, we can get $\mathrm{N}_B(t)$ and its activity $\mathrm{A}_B(t)$:

$$N_B(t) = \frac{\lambda_A}{\lambda_B - \lambda_A} N_0 \left(e^{-\lambda_A t} - e^{-\lambda_B t}\right) \qquad (2)$$

$$A_B(t) = \lambda_B N_B(t) = \frac{\lambda_A \lambda_B}{\lambda_B - \lambda_A} N_0 \left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$

How was equation (2) derived ?

We multiply both sides of the equation by $e^{\lambda_B t}$

$$\frac{dN_B(t)}{dt} = -\lambda_B N_B(t) + \lambda_A N_A \qquad \Big| \times e^{\lambda_B t}$$

and we rearrange to obtain

$$e^{\lambda_B t}\frac{dN_B(t)}{dt} + \lambda_B e^{\lambda_B t} N_B(t) = \lambda_A N_A e^{\lambda_B t}$$

This can be written as

$$\frac{d}{dt}(e^{\lambda_B t} N_B(t)) = \lambda_A N_A e^{\lambda_B t}$$

where we use $N_A(t) = N_0 e^{-\lambda_A t}$ to obtain the form

$$\frac{d}{dt}(e^{\lambda_B t} N_B(t)) = \lambda_A N_0 e^{(\lambda_B - \lambda_A)t}$$

How was eq.(2) derived (part II)

We multiply by *dt* and integrate both sides

$$\int_0^t d(e^{\lambda_B t} N_B(t)) = \int_0^t \lambda_A N_0 e^{(\lambda_B - \lambda_A)t} dt$$

to obtain

$$N_B(t) e^{\lambda_B t} - 0 = \frac{\lambda_A}{\lambda_B - \lambda_A} N_0 (e^{(\lambda_B - \lambda_A)t} - 1)$$

which gives us

$$\boxed{N_B(t) = \frac{\lambda_A}{\lambda_B - \lambda_A} N_0 (e^{-\lambda_A t} - e^{-\lambda_B t})}$$

QED

$A \rightarrow B \rightarrow C$ decay series

For the stable element C from such a series one would obtain:

$$N_C(t) = N_0 \left[1 - \frac{\lambda_B e^{-\lambda_A t} - \lambda_A e^{-\lambda_B t}}{\lambda_B - \lambda_A} \right]$$

which we derived using $N_0 = N_A(t) + N_B(t) + N_C(t)$

Instead we will focus on $N_B(t)$ and investigate a few special cases:

- $\lambda_A \gg \lambda_B$ (parent decays quickly)
- $\lambda_A = \lambda_B$
- $\lambda_A < \lambda_B$
- $\lambda_A \ll \lambda_B$ (parent is long lived)

$A \rightarrow B \rightarrow C$ decay series for $\lambda_A \gg \lambda_B$

Parent decays quickly, $\tau_A \ll \tau_B$

The number of daughter nuclei rises to maximum, then decays with constant λ_B.

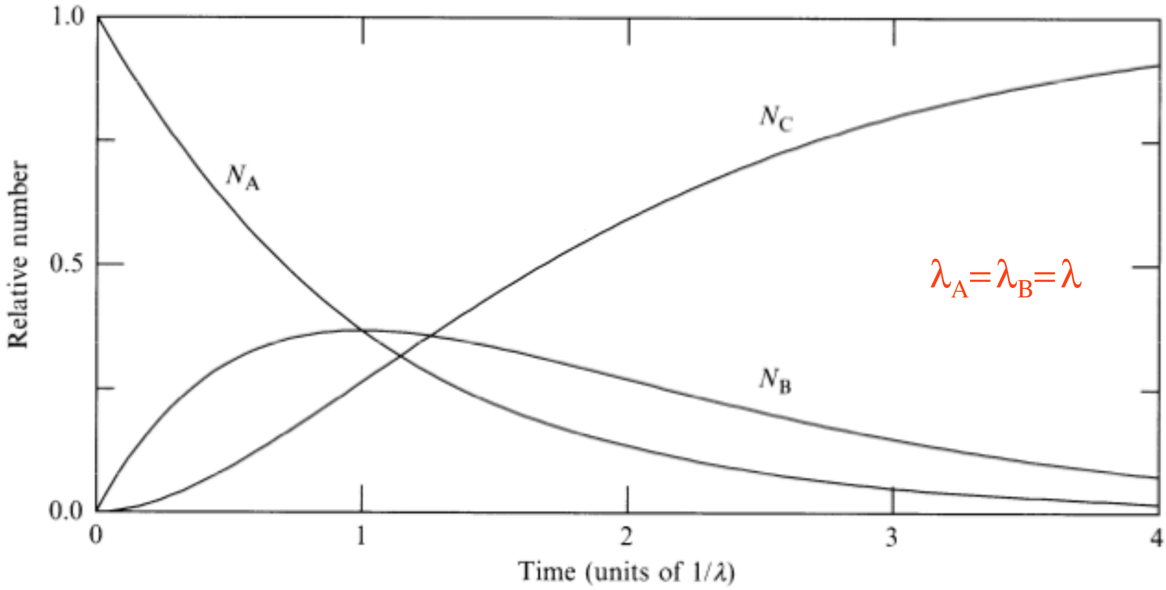
$\lambda_A \gg \lambda_B$

$$N_B(t) = \underbrace{\frac{\lambda_A}{\lambda_B - \lambda_A}}_{=-1} N_0 \left(\underbrace{e^{-\lambda_A t}}_{=0} - e^{-\lambda_B t} \right) \xrightarrow{\lambda_A \gg \lambda_B} N_0 e^{-\lambda_B t}$$

After a given time, daughter nuclei decay almost as if there were no parent nuclei.

A → B → C decay series for $\lambda_A = \lambda_B$

The solution of eq.(1) when $\lambda_A = \lambda_B = \lambda$ is: $N_B(t) = \lambda N_0 t e^{-\lambda t}$



A → B → C decay series for $\lambda_A < \lambda_B$

Parent A decays slower than the daughter B.

Ratio of activities becomes constant after a sufficiently long time:

$$\frac{A_B}{A_A} = \frac{\lambda_B N_B}{\lambda_A N_A} = \frac{\lambda_B}{\lambda_B - \lambda_A}\left(1 - e^{-(\lambda_B - \lambda_A)t}\right)$$

$$\approx \frac{\lambda_B}{\lambda_B - \lambda_A} \quad \text{when} \quad t \to \infty$$

A → B → C decay series for $\lambda_A \ll \lambda_B$

Parent nucleus is long lived: $\lambda_A \ll \lambda_B$ or $\tau_A \gg \tau_B$ so:

$$e^{-\lambda_A t} \approx 1 \;\Rightarrow\; N_A \approx N_0$$

$$\Rightarrow N_B \approx N_A \frac{\lambda_A}{\lambda_B}\left(1 - e^{-\lambda_B t}\right)$$

After a sufficiently long time $\left(1 - e^{-\lambda_B t}\right) \to 1$

$$\Rightarrow \lambda_A N_A = \lambda_B N_B \quad \Leftrightarrow \quad dN_B / dt = 0$$ in eq. (1)

Activity of A = Activity of B

This is known a *secular equilibrium*, i.e. at large times B is decaying at the same rate as it is produced.

Secular equilibrium ($\lambda_A \ll \lambda_B$)

An example of secular equilibrium is:

^{132}Te(12hrs) → ^{132}I(2.28hrs) → ^{132}Xe

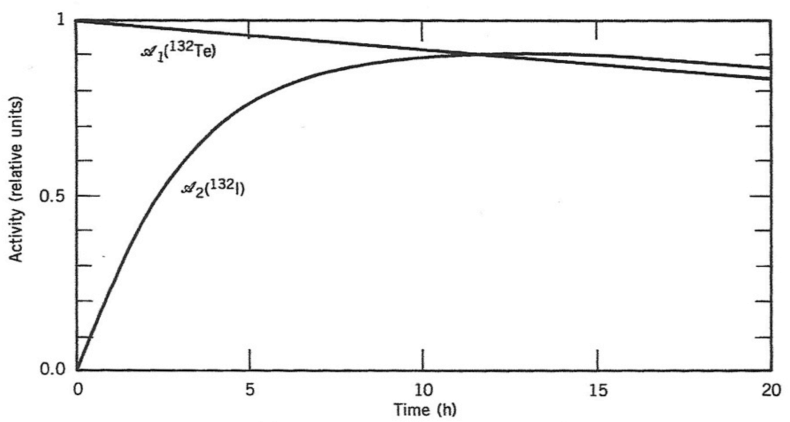


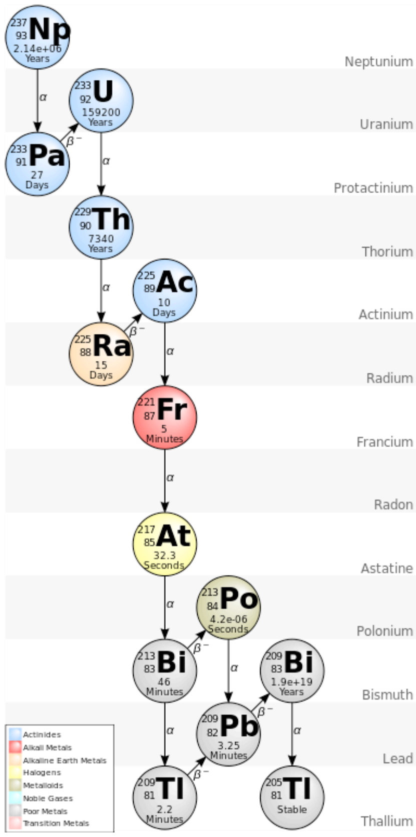
Figure 6.6 In the decay ^{132}Te (78 h) → ^{132}I (2.28 h) → ^{132}Xe, approximate secular equilibrium is reached at about 12 h.

Alpha decay chains

Because α-decay always decreases the atomic mass number A of the nucleus by 4, almost any decay will result in a nucleus with an atomic mass A' such that

$$A \bmod 4 = A' \bmod 4$$

As a result, there are four radioactive decay chains known as the Thorium (4n), Neptunium (4n +1), Radium (4n+2) and Actinium (4n+3) series.



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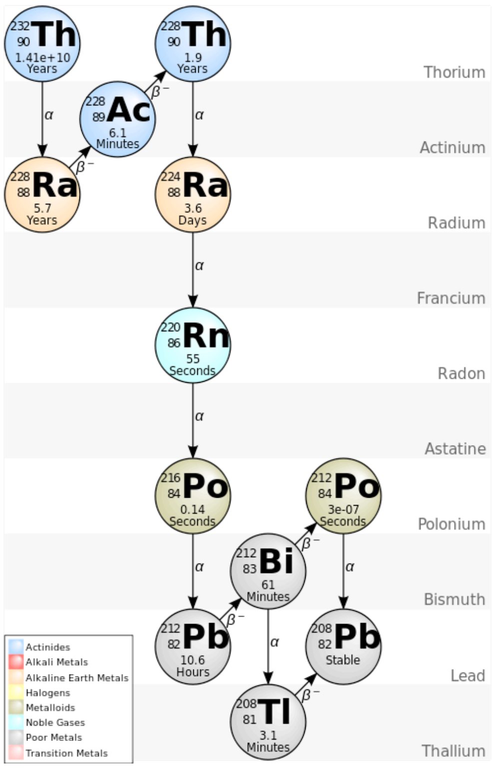
Thorium series and the age of the Earth



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^{232}Th has a very long half life ($t_{1/2}$ = 14Gyr) and goes through a long decay chain to stable ^{208}Pb.

It effectively behaves as if $^{232}\text{Th} \rightarrow {}^{232}\text{Pb}$

By measuring the relative abundance of ^{208}Pb:

$$\frac{N(^{208}Pb)}{N(^{232}Th)} = \frac{N_0(1-e^{-\lambda_{Th}t})}{N_0 e^{-\lambda_{Th}t}}$$

one can estimate of the age of the Earth at 4.54×10^9yr.

Radiometric dating

- Technique used to date geological materials (rocks) or man-made materials
- Based on a comparison between the observed abundance of a naturally occurring radioactive isotope and its decay products, using known decay rates.

Isotope		Half Life (years)	Useful Range (years)	Rock or mineral host
Parent	Daughter			
U-238	Pb-206	4.5×10^{9}	10^{7} - 4.6×10^{9}	Zircon Uraninite
K-40	A-40	1.3×10^{9}	5×10^{4} - 4.6×10^{9}	Micas Hornblende Volcanics
Rb-87	Sr-87	47×10^{9}	10^{7} - 4.6×10^{9}	Micas Orthoclase Igneous rocks
C-14	(N-14)	5730	100 - 7×10^{4}	Biol material CO_2

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Radiocarbon dating

- Carbon is a fundamental part of living tissue.
- There are 3 isotopes of carbon - ^{12}C, ^{13}C and ^{14}C - in the atmosphere, from where they are absorbed by living organisms.
  - The ratio of $^{14}C/^{12}C$ is known to be $\gamma_0 = 1.8\times10^{-12}$
  - ^{14}C is permanently created by cosmic rays, i.e. this isotopic ratio is constant in nature
- The concentration of ^{14}C in living organisms is the same as that in the environment
- When the organism dies it no longer absorbs ^{14}C. The ^{14}C in the organism decays but the amount of ^{12}C remains constant

$$^{14}C/^{12}C = \gamma = \gamma_0 e^{-\lambda t}$$

- By measuring the ratio of $^{14}C/^{12}C$ one can find out how much time has passed

$$t = \ln(\gamma_0/\gamma)/\lambda$$