Nuclear and Particle Physics

Lecture 9: Special Relativity II

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Transformation of space intervals in Special Relativity

We have discussed time dilation, where a time interval Δt_0 measured in the stationary frame is related to the time interval Δt measured in the moving frame by

$$\Delta t = \gamma \, \Delta t_0$$

Similarly with time dilation, the length of a ruler measured in a frame in which it is moving is less than the length measured in its rest frame

$$\Delta l = \frac{\Delta l_0}{\gamma} \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

This effect is called length contraction, also known as FitzGerald-Lorenz contraction.

Minkowski spacetime

Hermann Minkowski
(1864-1909)

- The mathematician Hermann Minkowski proposed in 1907 a four-dimensional interpretation of Special Relativity
- He introduced the unification of space and time into an inseparable 4D entity ('the World')
- The Lorentz geometry of Special Relativity can be elegantly represented in the 4D Minkowski spacetime

"The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

H. Minkowski - Address at at the 80th Assembly of German Natural Scientists and Physicians, Köln, Sep 1908

Four-vectors

- In three coordinates x, y, z we can define a position in space $\vec{x} = (x, y, z)$
- The distance d between two points can be defined as

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

- This distance is invariant under Galilean transformations

- In Relativity we need to deal with space *and* time
- The Minkowski spacetime (os simply *space*) is defined as a 4-dimensional vector space with 3 spacial and 1 temporal dimensions $X=(ct,x,y,z)$
- The Minkowski space replaces Euclidean space as the natural framework of reality once relativity comes into play
- The corresponding distance in Minkowski space is

$$S^2 = c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 \qquad (1)$$

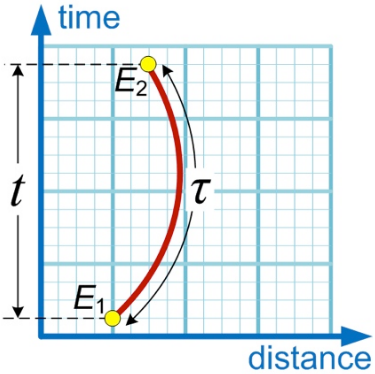
- This S^2 is invariant under Lorentz transformations, i.e. *Lorentz invariant*

4-velocity

- The position 4-vector in Minkowski space is defined as $X=(ct,x,y,z)$ so we can calculate

$$V=\frac{dX}{d\tau}=\gamma\frac{d}{dt}(ct,\vec{x})=\gamma(c,\frac{d\vec{x}}{dt})=\gamma(c,\vec{v})$$

where τ is the proper time, i.e. the elapsed time between two events as measured by a clock that passes through both events.



- In the rest frame:

$$\vec{v}=0,\gamma=1 \quad \Rightarrow \quad V=(c,0,0,0)$$

which can be interpreted to mean that when we are 'at rest' we actually move at maximum speed through time.

- Note that for light there is no rest frame and v defined above does not make sense

$$V=\gamma(c,\vec{c}) \quad \Rightarrow \quad V^2=\gamma(c^2-\vec{c}^2)=0$$

(see next slide)

Invariants

- The scalar product of two generic 4-vectors 'a' and 'b' is defined as:

$$ab=a_0b_0-a_1b_1-a_2b_2-a_3b_3=a_0b_0-\vec{a}\cdot\vec{b}$$

- This product is Lorentz invariant, *i.e.* it has the same value in all inertial frames.

- Analogous with the spacetime vector, one can define the 4-momentum as

$$P\equiv mV$$

where m is the rest mass and V is the 4-velocity constructed as

$$V=\gamma(c,\vec{v})$$

such that in terms of the total energy and the 3-momentum, we have

$$P=m\gamma(c,\vec{v})$$

$$\Rightarrow \boxed{P=(E/c,\vec{p})} \qquad (2)$$

Example

A particle of mass *M* is at rest when it splits into two fragments, each of rest mass *m*, which move with velocities *(v,0,0)* and *(-v,0,0)*. Show that $M=2m\gamma$.

By conservation of 4-momentum we have:

$$M(c,0,0,0) = m\gamma(c,v,0,0) + m\gamma(c,-v,0,0)$$

Hence, for the first component we have

$$Mc = m\gamma c + m\gamma c \quad \Rightarrow \quad M = 2m\gamma$$

Note that $M>2m$.

Invariant mass

- Applying the scalar product to the 4-momentum defined in (2), we can construct the *invariant*

$$P^2 = \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2 \qquad (3)$$

  where $m = \frac{\sqrt{P^2}}{c}$ is called the invariant mass, which for a particle is identical to its rest mass.

- Equation (3) can be also written in the form

$$\boxed{E^2 = \vec{p}^{\,2}c^2 + m^2c^4} \qquad (4)$$

- Note that since the 4-momentum squared is Lorentz invariant, one can choose an inertial system where $\vec{p} = 0$ and then $E = mc^2$.

How much energy ?

Let us calculate the ratio

$$\frac{E}{m} = c^2 = 9 \times 10^{16} m^2/s^2 = 9 \times 10^{16} J/kg$$

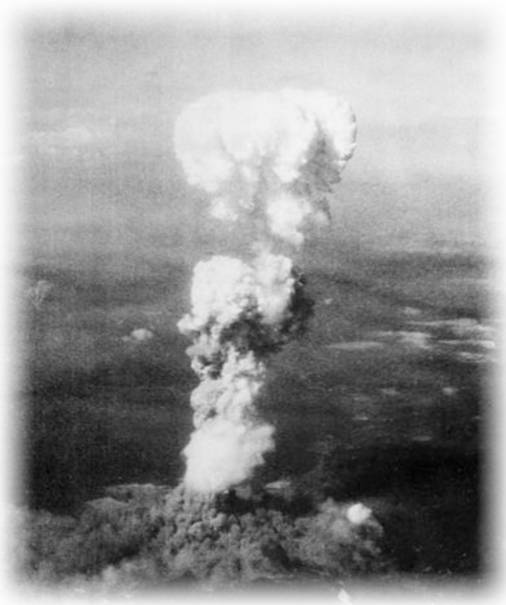
This means that $1g$ of mass is approximately equivalent to:

- 90 TJ (9 trillion Joules) or
- 25 GWh (25 million kilowatt-hour)
- 21.5 Tcal (billion kilocalories)
- 21 kt (kilotons TNT-equivalent energy)

Listen to 10 famous physicists explaining mass-energy equivalence at:
http://www.pbs.org/wgbh/nova/einstein/experts.html

Letters to Roosevelt

- Between 1939 and 1945, Einstein signed a series of letters to president Roosevelt about the possibility of constructing "*extremely powerful bombs of a new type*" including hints that the Germans might be already undertaking research in this direction.
- The first letter, dated August 2, 1939 is considered the letter that launched the nuclear arms race.
- Later on Einstein would take full responsibility for the consequences, calling it "the greatest mistake" of his life.



$E=mc^2$ and the Hiroshima bomb

- The energy released from a ^{235}U fission reaction is $Q \simeq 183\ MeV$
- In one kilogram of ^{235}U there are

  $\nu = 1000/235 = 4.255\ moles$
- One mole of any substance contains $N_A = 6.023\times10^{23}$ atoms.
- That means 1kg of ^{235}U contains

  $N = \nu N_A = 4.255 \times 6.023\times10^{23} = 2.56\times10^{24}\ nuclei$
- Say only a fraction $f = 0.8$ of these nuclei participate in the chain reaction. The total energy released is

  $Q_{tot} = f\,NQ = 0.8 \times 2.56\times10^{24} \times 183 \times 1.602\times10^{-1} = 60\ TJ$
- The mass transformed into energy here is

  $m = 6\times10^{13} / (3\times10^{8})^2 = 0.67\times10^{-3}\ kg \simeq 0.67\ g$!!

Relativistic mass

- In equation (2) we have introduced the relativistic momentum

$$\boxed{\vec{p} = \gamma m \vec{v}}$$

- The quantity $\gamma m = \dfrac{m}{\sqrt{1 - v^2/c^2}}$

  is the relativistic mass (the rest mass is often denoted by m_0).

- One should note that

$$\lim_{\beta\to1} \frac{m}{\sqrt{1-\beta^2}} = \infty$$

  which suggests that an object with non-zero rest mass can not reach the speed of light.

Forces and energy in relativity

- Newton's 2nd law:

$$F = \frac{dp}{dt} = \frac{d}{dt}\frac{mv}{\sqrt{1 - v^2/c^2}} \quad \Rightarrow \quad F = \frac{m}{\left(1 - v^2/c^2\right)^{3/2}} \quad \Rightarrow \quad a = \frac{F}{m}\left(1 - v^2/c^2\right)^{3/2}$$

- For increasing velocity the acceleration generated by a constant force decreases.

- The total energy is: $$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = K + mc^2$$

- The kinetic energy: $$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$$

Example

A pion is moving at at speed of $0.9c$. Let us calculate the momentum, total energy and the kinetic energies of the pion.

$$\beta = 0.9 \quad \Rightarrow \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = 2.29$$

So then

$$p = \gamma m_0 v = \gamma\beta m_0 = 2.29 \times 0.9 \times 139.6 MeV/c^2 = 228.2\, MeV/c$$

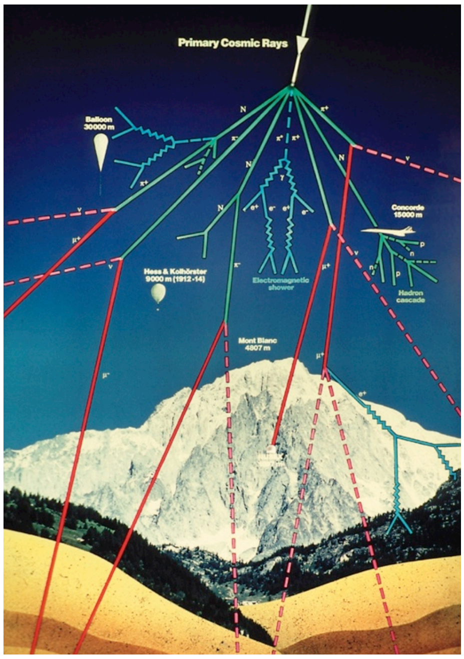
Energy:

$$E = \gamma m_0 c^2 = 2.29 \times 139.6 MeV/c^2 \approx 320\, MeV$$

Kinetic energy: $K = E - m_0 c^2 = 320 - 139.6 = 180.4\ MeV$

Example: Muon lifetime

- Cosmic ray muons are created when highly energetic radiation from deep space interacts with atoms in the Earth's atmosphere. The initial collisions create pions which decay into muons.
- 20 GeV muons are produced in the upper atmosphere, 15 km above sea level.
- The muon has a measured mean lifetime $\tau \simeq 2.2\ \mu s$
- If muons travel at a velocities close to the speed of light, will they reach ground level to de detected ?



Muon lifetime puzzle

A muon moving at the speed of light would travel the 15km distance in a time

$$t = H / c = 15000m / (0.998 \times 3 \times 10^{8} m / s) \approx 50 \mu s$$

so the fraction of muons reaching ground level is

$$\frac{N}{N_0} = \exp\left(-\frac{t}{\tau}\right) = \exp\left(-\frac{50}{2.2}\right) \simeq 1.3 \times 10^{-10}$$

i.e. practically nothing will be detected.

However, for an observer on Earth the lifetime of the muons is

$$\tau_E = \gamma\tau = (E / m_0 c^2)\tau = (20GeV/106MeV)\tau \approx 189\tau$$

and then the fraction of muons reaching the ground will be

$$\frac{N}{N_0} = \exp\left(-\frac{t}{\tau_E}\right) = \exp\left(-\frac{50}{2.2 \times 189}\right) \simeq 0.89 \simeq 90\%$$

Muon lifetime puzzle cont'd

Another way of looking at this problem:- From the muon's reference frame, the 15km distance between upper atmosphere and ground is contracted by a γ factor:

$$H_\mu = \frac{H}{\gamma} = \frac{15}{189} km = 0.0793 km \approx 79m$$

and the time needed to travel this distance is

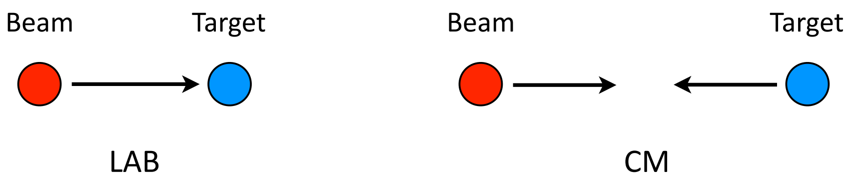
$$t_\mu = H_\mu / c = 79m / 3\times10^8 m/s \approx 0.26\mu s$$

The fraction of muons left undecayed by the time they reach the ground level is

$$\frac{N}{N_0} = \exp\left(-\frac{t_\mu}{\tau}\right) = \exp\left(-\frac{0.26}{2.2}\right) \simeq 0.89 \simeq 90\%$$

Frames of reference

- The two most commonly used frames of reference for particle kinematics are the laboratory system (LAB) and the centre of mass system (CM).



Fixed target

- In the LAB we have: $P_B = (E_B/c,\ \vec{p}_B)$ $\quad P_T = (m_T c,\ \vec{0})$
- In CM: $P_B = (E_B^*/c,\ \vec{p}_B^*)$ $\quad P_T = (E_T^*/c,\ \vec{p}_T^*)$ $\quad \vec{p}_B^* + \vec{p}_T^* = 0$
- The invariant mass squared of the system is

$$s \equiv (P_B + P_T)^2 / c^2 = (E_B + E_T)^2 / c^4 - (\vec{p}_B + \vec{p}_T)^2 / c^2$$

Example: LHC protons

The LHC collides 7TeV protons. What is the invariant mass of the 2-proton system ?

Beam (1) ⟶ ⟵ (2) Beam CM

We have 2TeV protons colliding head to head:

$$P_1 = (E/c, \vec{p}) \qquad P_2 = (E/c, -\vec{p})$$

Then

$$s = M_{2p}^2 = (P_1 + P_2)^2/c^2 =$$

$$= (E+E)^2/c^4 - (\underbrace{\vec{p} - \vec{p}}_{=0})/c^2 = \left(2E/c^2\right)^2$$

i.e. $M_{2p} = 14TeV/c^2$

Example: proton-hydrogen collisions in the atmosphere

For cosmic ray protons collide with hydrogen nuclei in the atmosphere, what energy of the protons will produce the same CM energy as the LHC ?

Beam (1) ⟶ (2) Target LAB

We have a proton with energy E and momentum p colliding with a proton practically at rest:

$$P_1 = (E/c, \vec{p}) \qquad P_2 = (m_p c, 0)$$

with (3)

Then

$$s = M_{2p}^2 = (P_1 + P_2)^2/c^2 = (E + m_p c^2)^2/c^4 - \vec{p}^2/c^2 =$$

$$= 2m_p^2 + 2Em_p/c^2$$

very small

and

$$E = \frac{(M_{2p}^2 - 2m_p^2)}{2m_p c^2} = \frac{(14^2 - \cancel{2\times 0.001^2})}{2\times 0.001} = 98000\,TeV \;\; !!$$

The Large Hadron Collider

The LHC synchrotron is designed to collide opposing particle beams of either protons at up to 7 TeV per nucleon, or Pb nuclei at an energy of 574 TeV per nucleus (2.76 TeV per nucleon-pair).

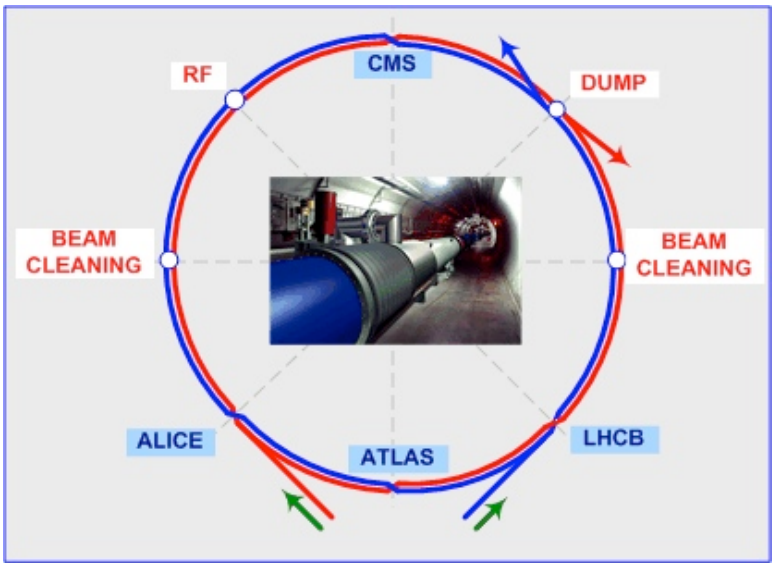


Image credits: cern.ch