Nuclear and Particle Physics Part 3: Radioactivity

Dr. Dan Protopopescu

Kelvin Building, room 524

Dan.Protopopescu@glasgow.ac.uk

Radioactivity

- Radioactivity is a natural process through which nuclei of unstable elements radiate excess energy in the form of particles
- The underlaying process is called *radioactive decay*
- Radioactive decay is:
  - *spontaneous* - occurs without any interaction* with other atomic constituents
  - a *stochastic* process at the level of single atoms, in that it is impossible to predict when a given nucleus will decay

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* except for decays via *electron capture* or *internal conversion*, when an inner electron of the radioactive atom is involved in the process

The law of radioactive decay

- If a sample of material contains N radioactive nuclei then the number decaying, *dN*, in a time *dt* will be proportional to *N*
- A quantity that decreases at a rate proportional to its value is said to be subject to exponential decay
- N_0 is the number of nuclei at time t=0 and *N(t)* is the number of nuclei that *have not* decayed by time *t*

$$\frac{dN}{dt} = -\lambda N$$

$$\lambda = -\frac{dN/dt}{N}$$

λ is the decay constant defined as the probability per unit time that a nucleus will decay

$$N(t) = N_0 e^{-\lambda t}$$

How was that derived ?

The number dN of nuclei decaying in a time interval dt will be proportional to N. Mathematically, this is written as (1):

$$\frac{dN}{dt} \propto N$$

If I introduce a constant λ and add a minus sign to take into account the fact that N decreases in time, this can be rewritten as (2):

$$\frac{dN}{dt} = -\lambda N$$

This can be re-arranged as (3):

$$\frac{dN}{N} = -\lambda dt$$

Derivation continued ...

At the time t=0 we have started with N_0 nuclei, so we integrate this with the limits

$$\int_{N_0}^{N} \frac{dN}{N} = -\int_{0}^{t} \lambda dt$$

which gives (4)

$$\ln \frac{N}{N_0} = -\lambda t$$

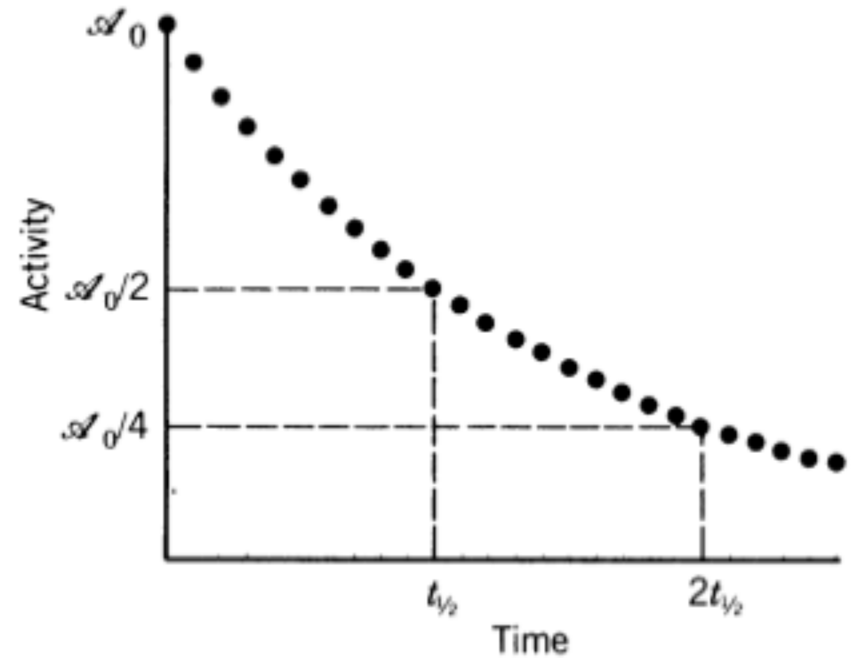
Eq. (4) can be written as

$$N = N_0 e^{-\lambda t}$$

or, if I want to explicitly indicate that N is a <u>function</u> of time

$$N(t) = N_0 e^{-\lambda t}$$

Decay rate and activity



- It is experimentally difficult to directly measure the number of nuclei that have not decayed
- It is more straightforward to measure the activity, *A(t)* of a sample, defined as the number of nuclei decaying per unit time (e.g. as clicks or counts from a Geiger counter in a given time)
- Activity will also follow the exponential decay law with A_0= initial activity = λN_0
- This assumes that we measure over a time *t* that is short compared to $1/\lambda$ ($t \ll 1/\lambda$)

$$A(t) = -\frac{dN(t)}{dt} = \lambda N(t)$$
$$= \lambda N_0 e^{-\lambda t}$$
$$= A_0 e^{-\lambda t}$$

Units of activity

- SI unit of activity is the becquerel (Bq)
  1 Bq = 1 decay/second
- Old unit: the curie (the activity of 1g of radium isotope ^{226}Ra)
  1 Ci = 3.7×10^{10} Bq = 37 GBq
- No account is taken of the type of radiation or how much energy the decay products have

Some examples	
Activity of the radioisotope (e.g. ^{137}Cs or ^{60}Co) in a radiotherapy machine	1000 Ci
Activity of the naturally occurring ^{40}K in the human body	0.1 µCi

Decay probability

- If there are N_0 nuclei at the time t, then the number decaying per unit time between t and $t+dt$ is:

$$-\frac{dN}{dt} = \lambda N_0 e^{-\lambda t}$$

- The probability of a single nucleus decaying in the time interval dt is then given by:

$$P_{decay}(t)dt = \frac{1}{N_0}\lambda N_0 e^{-\lambda t}dt = \lambda e^{-\lambda t}dt$$

Mean lifetime τ

In general the *mean* of a variable x that is distributed according to f(x) is given by:

$$\bar{x} = \frac{\int x f(x)\,dx}{\int f(x)\,dx}$$

To determine the mean life i.e. the mean time until an unstable nucleus decays we apply:

$$\bar{t} = \tau = \frac{\int_0^\infty t\lambda e^{-\lambda t}\,dt}{\int_0^\infty \lambda e^{-\lambda t}\,dt} = \frac{1}{\lambda}$$

$$N(t) = N_0 e^{-\lambda t} = N_0 e^{-t/\tau}$$

$$A(t) = A_0 e^{-\lambda t} = A_0 e^{-t/\tau}$$

The mean lifetime τ of the nucleus is the inverse of the decay constant λ.

Fraction surviving after 1 mean lifetime = e^{-1} = 0.37

after 2 mean lifetimes = e^{-2} = 0.135 etc.

Half-life $t_{1/2}$

- The half-life is the time after which half the sample has decayed:

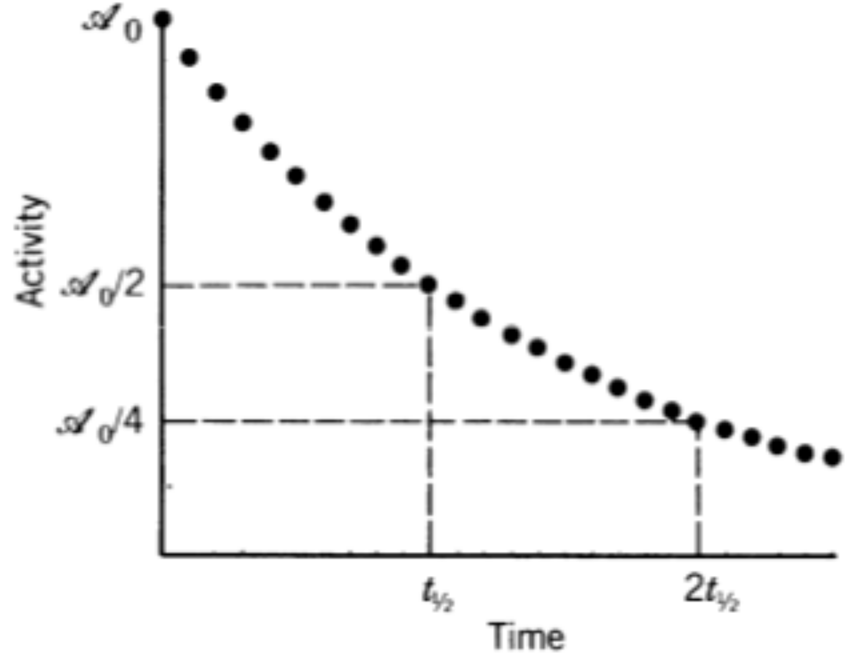
$$\text{When } N = \frac{N_0}{2} \quad t = t_{1/2}$$

$$\frac{N_0}{2} = N_0 e^{-\lambda t_{1/2}}$$

$$\Rightarrow e^{\lambda t_{1/2}} = 2$$

$$\Rightarrow \lambda t_{1/2} = \ln(2)$$

$$\Rightarrow t_{1/2} = \frac{\ln 2}{\lambda} = \tau \ln 2$$



Half-life < mean lifetime
Fraction surviving N half-lives=2^{-N}

Recap

Concept	Equation	Definition
Exponential decay	$N(t) = N_0 e^{-\lambda t}$	Number of nuclei that have not decayed by time *t*
Activity	$A(t) = \lambda N_0 e^{-\lambda t}$	Number of nuclei decaying per unit time, where $\lambda N_0 = A_0$
Decay probability	$P_{decay}(t) = \lambda e^{-\lambda t}$	Probability of a single nucleus decaying in the interval $t \rightarrow t+dt$
Mean lifetime	$\tau = 1/\lambda$	Mean time until an unstable nucleus decays
Half-life	$t_{1/2} = \ln 2/\lambda$	Time after which half the radioactive sample has decayed

Nuclear and Particle Physics
Lecture 4

Dr. Dan Protopopescu

Kelvin Building, room 524

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Recap puzzle

Concept	Equation	Definition
Exponential decay	$N(t) = N_0 e^{-\lambda t}$	Number of nuclei that have decayed in the time *t*
Activity	$A(t) = A_0 e^{-\lambda t}$	Number of nuclei decaying per unit time
Decay probability	$P_{decay}(t) = \lambda e^{-\lambda t}$	Probability of a single nucleus decaying in the interval *t*
Lifetime	$\tau = 1/\lambda$	Maximum time until an unstable nucleus decays
Half-life	$t_{1/2} = \ln 2/\lambda$	Time by which half the radioactive sample has not yet decayed

Answers

Concept	Equation	Definition
Exponential decay	$N(t) = N_0 e^{-\lambda t}$	Number of nuclei that have not decayed by time *t*
Activity	$A(t) = A_0 e^{-\lambda t}$	Number of nuclei decaying per unit time, where $A_0 = \lambda N_0$
Decay probability	$P_{decay}(t) = \lambda e^{-\lambda t}$	Probability of a single nucleus decaying in the interval $t \rightarrow t+dt$
Mean lifetime or simply *lifetime*	$\tau = 1/\lambda$	Mean time until an unstable nucleus decays
Half-life	$t_{1/2} = \ln 2/\lambda$	Time after which half the radioactive sample has decayed

Simple decay

If a sample of material consists of nucleus A which is unstable and decays to nucleus B (of which there are initially none) we have simply:

A ⟶ B

Nomenclature:

A - "parent"

B - "daughter"

The initial number of each nucleus is:

$N_A(t=0) = N_0$ (= total number of nuclei)

$N_B(t=0) = 0$

As nucleus A decays into nucleus B

$N_A(t) = N_0 e^{-\lambda_A t}$

and since

$N_0 = N_A(t) + N_B(t)$

$N_B(t) = N_0(1 - e^{-\lambda_A t})$

Alternative decay modes

An initial nuclide A that decays into two products: $A \rightarrow B + C$

We have at any time *t*: $N_A(t) + N_B(t) + N_C(t) = N_0$ and

$$\frac{dN_A}{dt} = -\lambda_A N_A, \quad \frac{dN_B}{dt} = \lambda_B N_A, \quad \frac{dN_C}{dt} = \lambda_C N_A$$

with $\lambda_A = \lambda_B + \lambda_C$. The decay constants λ_B and λ_C only determine the probabilities of the decays to products B or C

$$N_B(t) = \frac{\lambda_B}{\lambda_A} N_0 (1 - e^{-\lambda_A t})$$
$$N_C(t) = \frac{\lambda_C}{\lambda_A} N_0 (1 - e^{-\lambda_A t})$$

and

$$N_A(t) = N_0 - N_B(t) - N_C(t) = N_0 e^{-\lambda_A t}$$

Decay series (or chains)

Many heavy nuclei decay via complicated series involving several α and β decays. Consider the simple case of A → B → C, where C is stable and only A is present initially:

A → B → C

The number of nuclei A vary according to:

$$N_A(t) = N_o e^{-\lambda_A t}$$

The number of nuclei B as a function of time can be found from:

$$\frac{dN_B(t)}{dt} = -\lambda_B N_B(t) + \lambda_A N_A(t) \qquad (1)$$

where the first term is the decay of nuclei B and the second term is due to B being created from the decay of A.

Integrating, we can get $\mathrm{N}_B(t)$ and its activity $\mathrm{A}_B(t)$:

$$N_B(t) = \frac{\lambda_A}{\lambda_B - \lambda_A} N_0 \left(e^{-\lambda_A t} - e^{-\lambda_B t}\right) \qquad (2)$$

$$A_B(t) = \lambda_B N_B(t) = \frac{\lambda_A \lambda_B}{\lambda_B - \lambda_A} N_0 \left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$

How was equation (2) derived ?

We multiply both sides of the equation by $e^{\lambda_B t}$

$$\frac{dN_B(t)}{dt} = -\lambda_B N_B(t) + \lambda_A N_A \qquad \Big| \times e^{\lambda_B t}$$

and we rearrange to obtain

$$e^{\lambda_B t}\frac{dN_B(t)}{dt} + \lambda_B e^{\lambda_B t} N_B(t) = \lambda_A N_A e^{\lambda_B t}$$

This can be written as

$$\frac{d}{dt}(e^{\lambda_B t} N_B(t)) = \lambda_A N_A e^{\lambda_B t}$$

where we use $N_A(t) = N_0 e^{-\lambda_A t}$ to obtain the form

$$\frac{d}{dt}(e^{\lambda_B t} N_B(t)) = \lambda_A N_0 e^{(\lambda_B - \lambda_A)t}$$

How was eq.(2) derived (part II)

We multiply by *dt* and integrate both sides

$$\int_0^t d(e^{\lambda_B t} N_B(t)) = \int_0^t \lambda_A N_0 e^{(\lambda_B - \lambda_A)t} dt$$

to obtain

$$N_B(t) e^{\lambda_B t} - 0 = \frac{\lambda_A}{\lambda_B - \lambda_A} N_0 (e^{(\lambda_B - \lambda_A)t} - 1)$$

which gives us

$$N_B(t) = \frac{\lambda_A}{\lambda_B - \lambda_A} N_0 (e^{-\lambda_A t} - e^{-\lambda_B t})$$

QED

$A \rightarrow B \rightarrow C$ decay series

For the stable element C from such a series one would obtain:

$$N_C(t) = N_0 \left[1 - \frac{\lambda_B e^{-\lambda_A t} - \lambda_A e^{-\lambda_B t}}{\lambda_B - \lambda_A} \right]$$

which we derived using $N_0 = N_A(t) + N_B(t) + N_C(t)$

Instead we will focus on $N_B(t)$ and investigate a few special cases:

- $\lambda_A \gg \lambda_B$ (parent decays quickly)
- $\lambda_A = \lambda_B$
- $\lambda_A < \lambda_B$
- $\lambda_A \ll \lambda_B$ (parent is long lived)

$A \rightarrow B \rightarrow C$ decay series for $\lambda_A \gg \lambda_B$

Parent decays quickly, $\tau_A \ll \tau_B$

The number of daughter nuclei rises to maximum, then decays with constant λ_B.

$\lambda_A \gg \lambda_B$

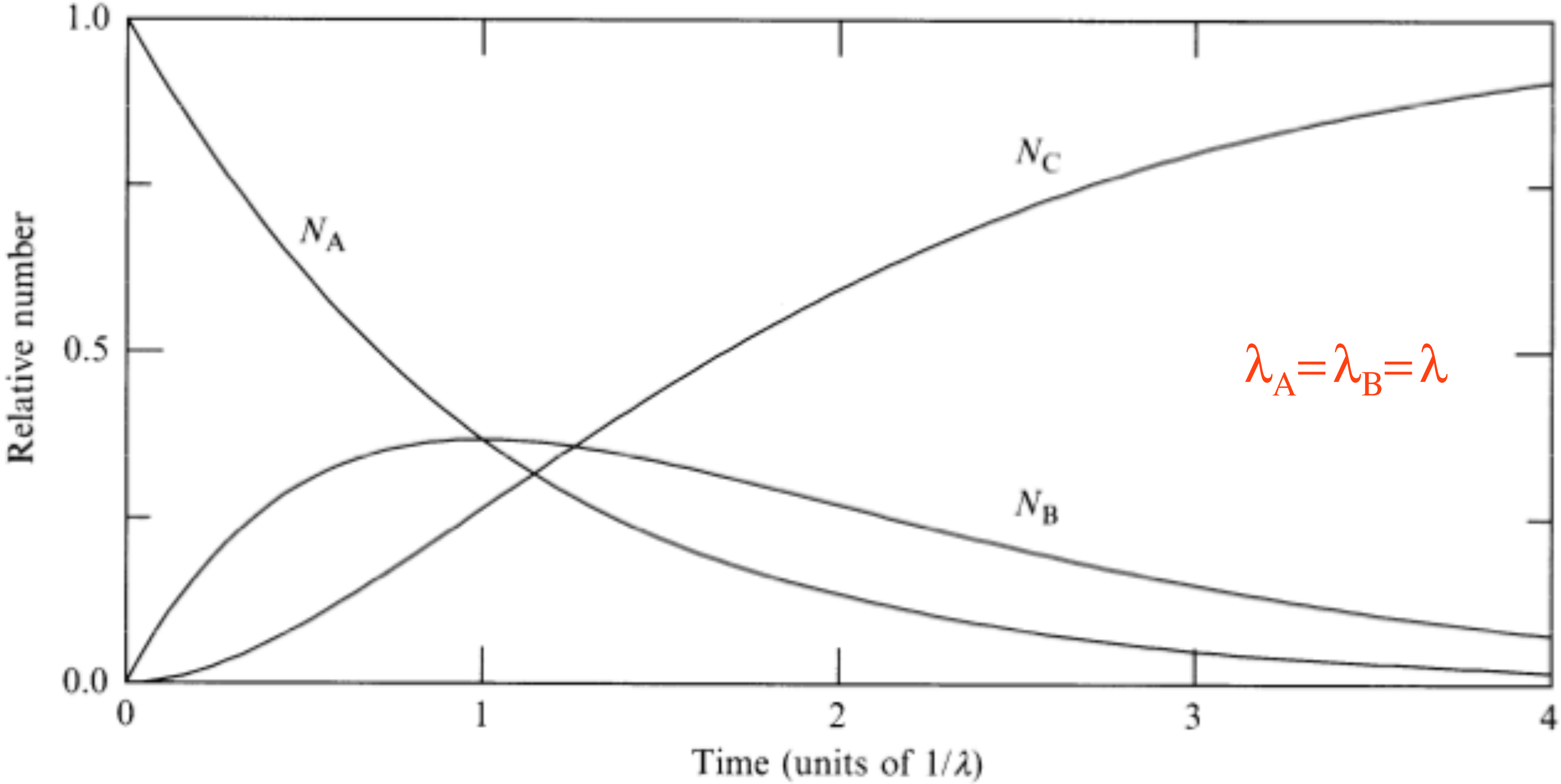
$$N_B(t) = \frac{\lambda_A}{\lambda_B - \lambda_A} N_0 (e^{-\lambda_A t} - e^{-\lambda_B t}) \xrightarrow{\lambda_A \gg \lambda_B} N_0 e^{-\lambda_B t}$$

=-1 =0

After a given time, daughter nuclei decay almost as if there were no parent nuclei.

$A \rightarrow B \rightarrow C$ decay series for $\lambda_A = \lambda_B$

The solution of eq.(1) when $\lambda_A = \lambda_B = \lambda$ is: $N_B(t) = \lambda N_0 t e^{-\lambda t}$



A → B → C decay series for $\lambda_A < \lambda_B$

Parent A decays slower than the daughter B.

Ratio of activities becomes constant after a sufficiently long time:

$$\frac{A_B}{A_A} = \frac{\lambda_B N_B}{\lambda_A N_A} = \frac{\lambda_B}{\lambda_B - \lambda_A}\left(1 - e^{-(\lambda_B - \lambda_A)t}\right)$$

$$\approx \frac{\lambda_B}{\lambda_B - \lambda_A} \quad \text{when} \quad t \to \infty$$

$A \rightarrow B \rightarrow C$ decay series for $\lambda_A \ll \lambda_B$

Parent nucleus is long lived: $\lambda_A \ll \lambda_B$ or $\tau_A \gg \tau_B$ so:

$$e^{-\lambda_A t} \approx 1 \;\Rightarrow\; N_A \approx N_0$$

$$\Rightarrow N_B \approx N_A \frac{\lambda_A}{\lambda_B}\left(1 - e^{-\lambda_B t}\right)$$

After a sufficiently long time $\left(1 - e^{-\lambda_B t}\right) \rightarrow 1$

$$\Rightarrow \lambda_A N_A = \lambda_B N_B \quad \Leftrightarrow \quad dN_B / dt = 0$$ in eq. (1)

Activity of A = Activity of B

This is known a *secular equilibrium*, i.e. at large times B is decaying at the same rate as it is produced.

Secular equilibrium ($\lambda_A \ll \lambda_B$)

An example of secular equilibrium is:

$^{132}Te(12hrs) \rightarrow \quad ^{132}I(2.28hrs) \rightarrow \quad ^{132}Xe$

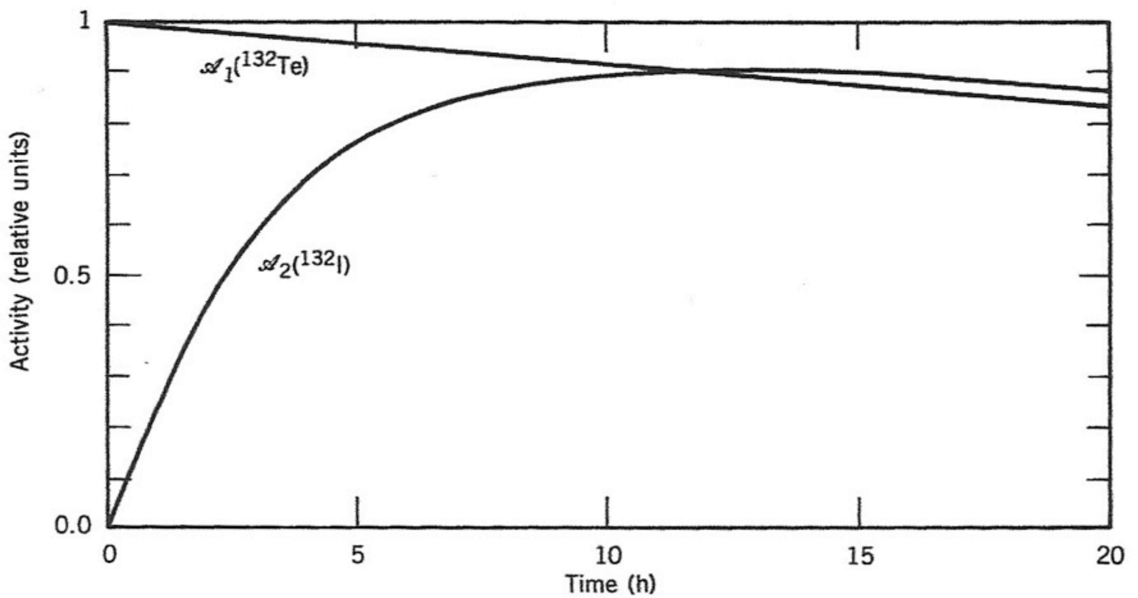


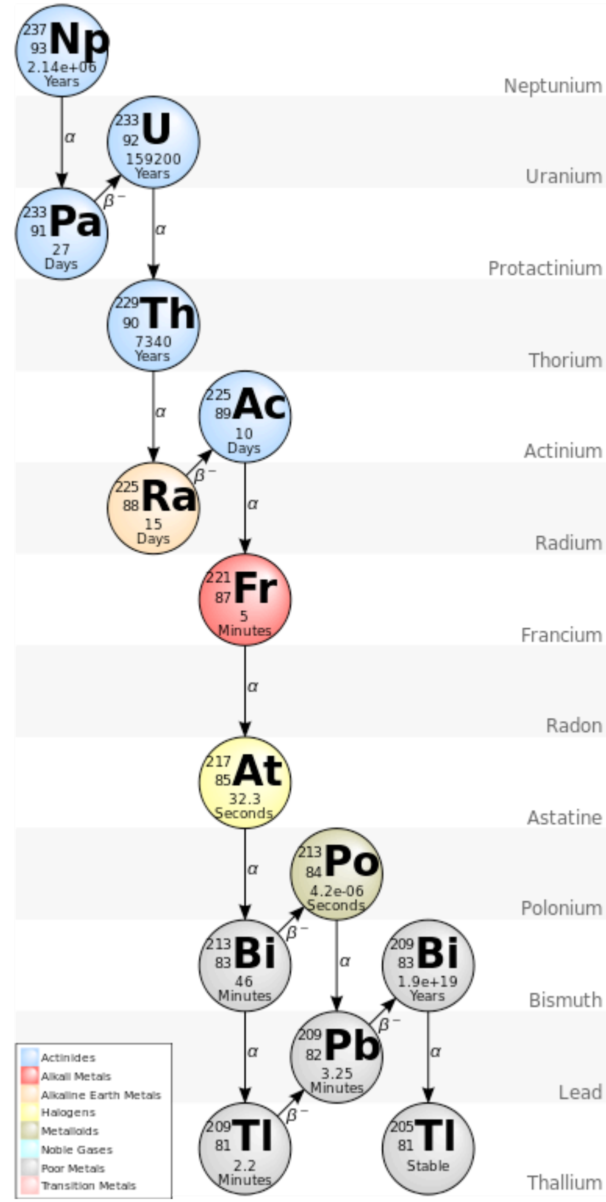
Figure 6.6 In the decay ^{132}Te (78 h) $\rightarrow$ ^{132}I (2.28 h) $\rightarrow$ ^{132}Xe, approximate secular equilibrium is reached at about 12 h.

Alpha decay chains

Because α-decay always decreases the atomic mass number A of the nucleus by 4, almost any decay will result in a nucleus with an atomic mass A' such that

$$A \bmod 4 = A' \bmod 4$$

As a result, there are four radioactive decay chains known as the Thorium (4n), Neptunium (4n +1), Radium (4n+2) and Actinium (4n+3) series.



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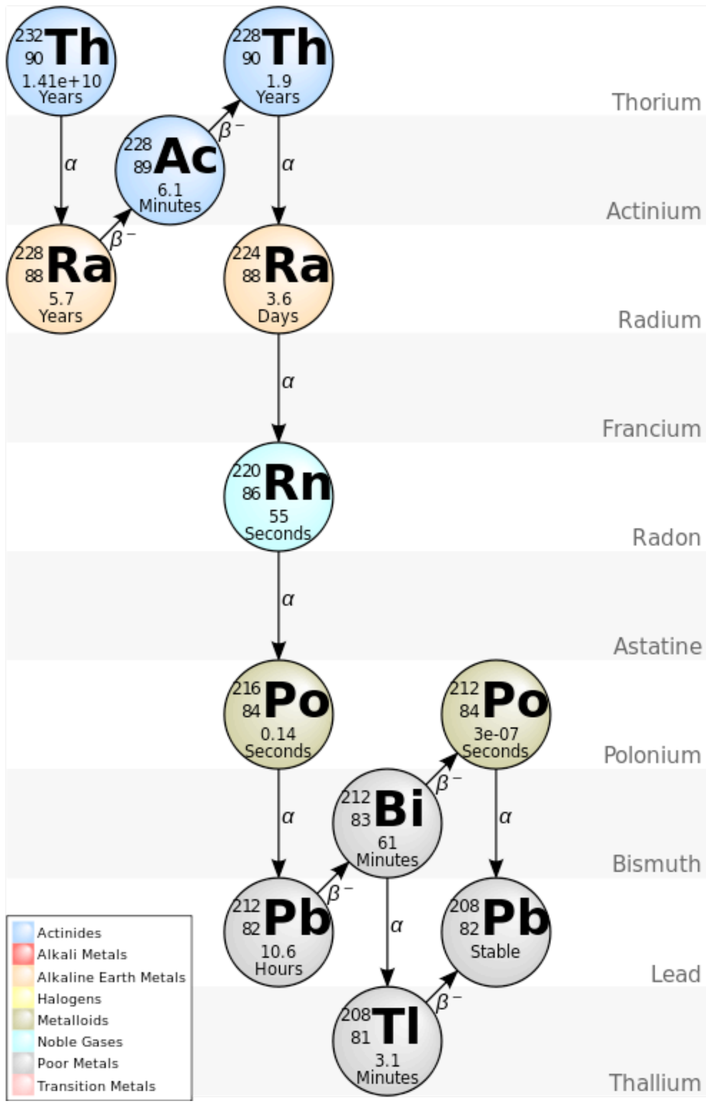
Thorium series and the age of the Earth



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^{232}Th has a very long half life ($t_{1/2}$ = 14Gyr) and goes through a long decay chain to stable ^{208}Pb.

It effectively behaves as if $^{232}\text{Th} \rightarrow {}^{232}\text{Pb}$

By measuring the relative abundance of ^{208}Pb:

$$\frac{N(^{208}Pb)}{N(^{232}Th)} = \frac{N_0(1-e^{-\lambda_{Th}t})}{N_0 e^{-\lambda_{Th}t}}$$

one can estimate of the age of the Earth at 4.54×10^9yr.

Radiometric dating

- Technique used to date geological materials (rocks) or man-made materials
- Based on a comparison between the observed abundance of a naturally occurring radioactive isotope and its decay products, using known decay rates.

Isotope Parent	*Isotope* Daughter	*Half Life (years)*	*Useful Range (years)*	*Rock or mineral host*
U-238	Pb-206	4.5×10^9	10^7 - 4.6×10^9	Zircon Uraninite
K-40	A-40	1.3×10^9	5×10^4 - 4.6×10^9	Micas Hornblende Volcanics
Rb-87	Sr-87	47×10^9	10^7 - 4.6×10^9	Micas Orthoclase Igneous rocks
C-14	(N-14)	5730	100 - 7×10^4	Biol material CO_2

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Radiocarbon dating

- Carbon is a fundamental part of living tissue.
- There are 3 isotopes of carbon - ^{12}C, ^{13}C and ^{14}C - in the atmosphere, from where they are absorbed by living organisms.
  - The ratio of $^{14}C/^{12}C$ is known to be $\gamma_0 = 1.8\times10^{-12}$
  - ^{14}C is permanently created by cosmic rays, i.e. this isotopic ratio is constant in nature
- The concentration of ^{14}C in living organisms is the same as that in the environment
- When the organism dies it no longer absorbs ^{14}C. The ^{14}C in the organism decays but the amount of ^{12}C remains constant

$$^{14}C/^{12}C = \gamma = \gamma_0 e^{-\lambda t}$$

- By measuring the ratio of $^{14}C/^{12}C$ one can find out how much time has passed

$$t = \ln(\gamma_0/\gamma)/\lambda$$